ON INJECTIONS AND SURJECTIONS OF CONTINUOUS FUNCTION SPACES

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ABSTRACT

It is shown that a linear isometry of $C(S)$ into $C(T)$ is generated by a linear extension from S to a quotient space of T. A dual theorem is proved for a quotient map of $C(S)$ onto $C(T)$. An application of the first theorem and a modification of a theorem by Banach and Mazur provide a negative answer to a problem posed by Pelczynski.

1. Introduction

If S is a compact Hausdorff space, $C(S)$ denotes the Banach space of all continuous real-valued functions on S with the supremum norm. 1_S denotes the constant function 1 on S. A linear $u: C(S) \to C(T)$ (S, T compact Hausdorff) is called *regular* if $\| u \| = 1$ and $u 1_S = 1_T$—this is equivalent to $u 1_S = 1_T$ and $u \geqq 0$. If $\pi: T \to S$ is continuous, then π^0 defined by $(\pi^0 f)(t) = f(\pi t)$ is a regular linear (even multiplicative) operator from $C(S)$ to $C(T)$. If π maps T *onto* S, then π^0 is a regular isometric embedding of $C(S)$ into $C(T)$. If π is a homeomorphic embedding of T into S, then the restriction map π^0 maps $C(S)$ onto $C(T)$ and $\| g \| = \min\{ \| f \| ;\ f \in C(S), \pi^0 f = g\}$ for every $g \in C(T)$.

The well-known Banach-Stone theorem ([6], p. 442) states that the most general linear isometry of $C(S)$ *onto* $C(T)$ is of the form $(\phi f)(t) = \alpha(t)(h^0 f)(t)$, where $\alpha \in C(T)$ satisfies $|\alpha(t)| = 1$ for all $t \in T$ and h is a homeomorphism of T onto S. If ϕ is regular, then obviously $\alpha(t) = 1$ and $\phi = h^0$. Holsztynski ([9] cf. also Pelczynski [11] and Tomašek [15]) obtained: If ϕ is a linear isometry from $C(S)$ to $C(T)$, then S is a quotient qZ of a subspace Z of T, and there is $\gamma \in C(Z)$ with $|\gamma| = 1$ and such that $(\phi f)(z) = \gamma(z)(q^0 f)(z)$ for all $z \in Z$, $f \in C(S)$. Unlike

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the Banach-Stone condition, Holsztynski's condition is not sufficient: $T=\beta N$, $S = Z = \beta N\backslash N$, q = the identity and $\gamma = 1$ satisfy the conditions, but $C(\beta N\backslash N)$ is not even isomorphic to a subspace of $C(\beta N)$ (Proposition 5).

Besides the isometric embeddings π^0: $C(S)\to C(T)$ which arise when S is a quotient space πT of T, there are also isometric embeddings $u: C(S)\to C(T)$ which arise when S is homeomorphic to a subspace S of T—these are the *norm-preserving linear extensions*, i.e. those u satisfying i^0u = identity of $C(S)$. Norm-preserving linear extensions always exist if S is metrizable [10] but may fail to exist in the general case (e.g. the case mentioned above: $T = \beta N$, $S = \beta N\backslash N$). An improvement of Holsztynski's theorem, Theorem 1, shows that $C(S)$ is isometric to a subspace of $C(T)$ if, and only if, there is a norm-preserving linear extension v: $C(S)\to C(Y)$, where $Y = \pi T$ is a quotient space of T, and that every regular isometric embedding of $C(S)$ into $C(T)$ is of the form $\pi^0 v$. The improvement is by showing the existence of a norm-preserving linear extension from the subspace $q^0C(S)$ of $C(Z)$ into $C(T)$.

A corollary of Theorem 1 and a natural modification of a classical result of Banach and Mazur enable us to answer in the negative a problem posed by Pelczynski [12].

In Sec. 3 we treat the dual problem — characterize all linear mappings τ of $C(S)$ onto $C(T)$ which, like the restrictions π^0 in the case when $\pi: T\to S$ is a homeomorphism, satisfy $\|g\| = \min\{\|f\|; f\in C(S), \tau f = g\}$. If $\pi: T\to S$ is an onto map, the same condition is satisfied by *norm-1 projections* of $C(S)$ onto $C(T)$, i.e. v: $C(S)\to C(T)$ satisfy $v\pi^0$ = the identity on $C(T)$. The general case is obtained by a combination of both.

2. Linear isometries from $C(S)$ to $C(T)$

Theorem 1 is an improvement of Holsztynski's theorem. The first part of the proof is essentially Holsztynski's proof and we repeat it for completeness sake.

THEOREM 1. *If ϕ: $C(S)\to C(T)$ is a linear isometry, then there are a continuous mapping π from T onto some Y, a homeomorphic embedding $j: S\to Y$, a norm-preserving linear extension v from $C(S)$ to $C(Y)$, and $\gamma\in\pi^0C(Y)$ with $|\gamma(\pi^{-1}jS)| = 1$ such that $\gamma\phi = \pi^0 v$. If ϕ is regular one can take v regular and $\gamma = 1_T$.*

PROOF. Define, for $s\in S$: $P(s) = \{f\in C(S);\ \|f\| = 1,\ f(t) = 1$ in a neighborhood of $s\}$, $\sigma(s) = \cap\{M(\phi f); f\in P(s)\}$ where $M(g) = \{t; |g(t)| = \|g\|\}$.

We show first that $\sigma \neq \emptyset$: If $f_1, \cdots, f_n \in P(s)$, then $1 = 1/n \sum_{i=1}^n \|f_i\| \geqq \|1/n \sum_{i=1}^n f_i\| \geqq |1/n \sum_{i=1}^n f_i(s)| = 1$, hence $\|1/n \sum_{i=1}^n \phi f_i\| = 1$ and for some $t_0 \in T$: $|1/n \sum_{i=1}^n \phi f_i(t_0)| = 1$. Since $|\phi f_i(t_0)| \leqq \|\phi f_i\| = \|f_i\| = 1$, this implies that $|\phi f_i(t_0)| = 1$ for $i = 1, \cdots, n$, i.e. $t_0 \in \bigcap_{i=1}^n M(\phi f_i)$. Thus, the family $\{M(\phi f_i); f \in P(s)\}$ has the finite intersection property, and by compactness — a non empty intersection.

If $s_1 \neq s_2$, take disjoint closed neighborhoods V_1, V_2, and $f \in C(S)$ satisfying $f(V_1) = 1 \geqq f \geqq 0 = f(V_2)$. Then $f \in P(s_1)$, $1 - f \in P(s_2)$. For $t \in \sigma s_1$ we have $|\phi f(t)| = |\phi 1(t)| = 1$, while for $t \in \sigma s_2$ we must have $|\phi 1(t) - \phi f(t)| = 1$. Since these requirements are incompatible (in the real case), $\sigma s_1 \cap \sigma s_2 = \emptyset$.

We show now that σK is closed for every closed $K \subset S$: Let $\{t_\alpha\}$ be a net in σK, with $t_\alpha \to t$. If $t_\alpha \in \sigma s_\alpha$, $s_\alpha \in K$ we can take, by compactness of K, a convergent subnet $s_\beta \to s \in K$. If $t \notin \sigma K$, then $t \notin \sigma s$ and there is a neighborhood V of $\sigma s = \cap \{M(\phi f); f \in P(s)\}$ (which is closed by definition) with $t \notin \bar{V}$. By compactness, V contains a finite intersection $\bigcap_{i=1}^n M(\phi f_i)$, where $f_i \in P(s)$. By the definition of $P(s)$, there are open neighborhoods W_i of s with $f_i(t) = 1$ for $t \in W_i$. Let $W = \bigcap_{i=1}^n W_i$. W is an open neighborhood of s, hence there is β_0 such that for all $\beta > \beta_0 : s_\beta \in W$, so that W is a neighborhood of s_β and $f_i \in P(s_\beta)$ for each i. Since $t_\beta \in \sigma s_\beta$, $t_\beta \in M(\phi f_i)$ for each i and therefore $t_\beta \in V$ for all $\beta > \beta_0$. Thus $t = \lim t_\beta \in \bar{V}$, which contradicts the choice of V.

In particular, $Z = \sigma S$ is a closed subset of T and the mapping σ^{-1} from Z onto S is continuous, hence S is homeomorphic to the quotient space Z/D, where D is the decomposition of Z into the sets σs.

Given any $f \in C(S)$ with $\|f\| = 1 = f(s)$, f can be uniformly approximated by elements of $P(s)$, so that for such f and any $t \in \sigma s$ we have $|\phi f(t)| = \lim_{P(s) \ni h \to f} |\phi h(t)| = 1$, i.e. $t \in M(\phi f)$. $|\phi 1_S| = 1$ on Z (since $1_S \in P(s)$ for all $s \in S$), hence replacing ϕ by $\psi = \gamma\phi$ where $\gamma \in C(T)$ satisfies $|\gamma| \leqq 1$, $\gamma(t) = \phi 1(t)$ on Z, we get a linear $\psi: C(S) \to C(T)$ which is still an isometry ($|(\psi f)(t)| = |\gamma(t)||\phi f(t)| \leqq \|\phi f\| \leqq \|f\|$, while if $\|f\| = 1$ then $|f(s)| = 1$ for some $s \in S$ and for $t \in \sigma s$ we have $|\psi f(t)| = |\gamma(t)||\phi f(t)| = |\phi f(t)| = 1$), $\psi 1_S = 1$ on Z, and for every $f \in C(S)$ with $0 \leqq f \leqq 1 = f(s)$ we have $(\psi f)(t) = 1$ for $t \in \sigma s$ (if $(\psi f)(t) = -1$, then $\psi(1 - f)(t) = 2$ which is impossible). By homogeneity, if $f \in C(S)$ satisfies $0 \leqq f \leqq f(s)$, then $(\psi f)(\sigma s) = f(s)$. If $f \in C(S)$ and $f \geqq 0 = f(s)$, then $\|f\| - (\psi f)(\sigma s) = \psi(\|f\| - f)(\sigma s) = (\|f\| - f)(s) = \|f\|$, hence $(\psi f)(\sigma s) = 0 = f(s)$.

If $0 \leqq f \in C(S)$ and s is any point in S, let $g = \min(f, f(s))$, then by the above $(\psi g)(\sigma s) = g(s) = f(s)$, while $\psi(f-g)(\sigma s) = 0 = (f-g)(s)$. So that $(\psi f)(\sigma s) = (\psi g)(\sigma s) + \psi(f-g)(\sigma s) = f(s)$. If $f \in C(S)$, $s \in S$ are any, let $g = \max(f, 0)$. Then $f(s) = g(s) - (g-f)(s) = \psi g(\sigma s) - \psi(g-f)(\sigma s) = \psi f(\sigma s)$. Therefore we have $i^0 \psi = (\sigma^{-1})^0$ (where $i: Z \to T$ is the embedding).

Let $\hat{D}$ be the trivial extension of D to a decomposition of T, i.e. $\hat{D}$ consists of the sets σs ($s \in S$) and the singletons $\{t\}$ ($t \in T \setminus Z$). Since D is closed in $\sigma S \times \sigma S$, it is closed in $T \times T$, and so is $\hat{D}$. Let $Y = T \setminus \hat{D}$, π — the quotient map. Y is compact, and $j(s) = \pi\{\sigma s\}$ defines a natural homeomorphic embedding of S into Y.

Since for every $f \in C(S)$, ψf is constant on the decomposition sets, $(vf)(y) = (\psi f)(\pi^{-1} y)$ is a well defined linear map from $C(S)$ into $C(Y)$ with norm $\leqq 1$. $(vf)(js) = (\psi f)(\pi^1 js) = (\psi f)(\sigma s) = f(s)$, i.e. v is a norm-preserving linear extension for j.

In conclusion we have $\gamma(t)\phi(t) = (\psi f)(t) = (\psi f)(\pi^{-1}\pi t) = (vf)(\pi t)$ $(\pi^0 vf)(t)$, i.e. $\gamma\phi = \pi^0 v$.

If $\phi 1 = 1$, we can take $\gamma = 1_T$, i.e. $\psi = \phi$, and get $(v1_S)(y) = 1_T(\pi^{-1} y) = 1$. Q.E.D.

Corollary 2. *If S is Stonian (i.e., an extremally disconnected compact) and $C(S)$ is isometric to a subspace of $C(T)$, then S is homeomorphic to a subspace Z of T admitting a linear extension of norm 1. If the isometry is regular, so is the extension.*

Proof. Let v, j, π, Y be as in Theorem 1. Since S is projective [8], there is a homeomorphic embedding $h: S \to T$ with $j = \pi^0 h$. $\pi^0 v$ is a linear extension for h $(h^0(\pi^0 v) = j^0 v = \mathrm{id}_{C(S)})$. Finally, if $\phi 1_S = 1_T$, $\pi^0 v 1_S = 1_T$. Q.E.D.

Pelczynski asked ([12] problem 17): Suppose $i: S \to T$ is a homeomorphic embedding, with T 0-dimensional, which admits a regular linear extension u. Is it a retract of T? The answer is in the negative — let S be the Yošida-Hewitt space [16], i.e. the maximal ideal space of $L^\infty[0,1]$ in its Gelfand topology, and $T = \beta N$. Then $C(S) = L^\infty[0,1] = L^1[0,1]^*$ is regularly isometric to a subspace of $m = C(T)$, as will be seen soon. By the former corollary there is a homeomorphism $i: S \to \beta N$ which admits a regular linear extension. On the other hand, iS cannot be a retract of βN, since S is not separable.

To establish the regular embedding of $C(S)$ in $C(T)$, we apply a simple modification of the classical result of Banach and Mazur [2].

Theorem 3. *If X is a separable (AL)-space* ([5], p. 100), *then there is a* *linear bounded* positive *operator T from l^1 onto X such that X is naturally isometric to $l^1/T^{-1}0$.*

PROOF. Let (x_n) be a dense sequence in the positive cone $P = \{x \in X; x \geqq 0\}$, $y_n = x_n/\|x_n\|$ (one may assume $x_n \neq 0$). For $\alpha \in l^1$, define $T_\alpha = T(\Sigma\alpha_n e_n) = \Sigma\alpha_n y_n$ — this last series converges absolutely, and $\|T_\alpha\| = \|\Sigma\alpha_n y_n\| \leqq \Sigma|\alpha_n| = \|\alpha\|$. Given $x \in P$, $\varepsilon > 0$, choose $x_{n_0} \in G(x_n)$ with $\|x - x_{n_0}\| < \varepsilon$, $x_{n_1} \in (x_n)$ with $\|x_{n_1}\| < \varepsilon$ and $\|x - x_{n_0} - x_{n_1}\| < \varepsilon/2$, $x_{n_2} \in (x_n)$ with $\|x - x_{n_0} - x_{n_1} - x_{n_2}\| < \varepsilon/4$ etc. Then $x = T\alpha$ where $\alpha = \Sigma_{i=0}^{\infty}\|x_{n_i}\| e_{n_i}$ satisfies $\|\alpha\| < \|x\| + 3\varepsilon$.

Given any $x \in X$, $\varepsilon > 0$ there are, by the above, $\alpha_+, \alpha_- \geqq 0$ in l^1 such that $T_{\alpha+} = x_+$, $T\alpha_- = x_-$, $\|\alpha_+\| < \|x_+\| + \varepsilon$, $\|\alpha_-\| < \|x_-\| + \varepsilon$. Then $\alpha = \alpha_+ - \alpha_-$ satisfies $T_\alpha = x$, $\|\alpha\| \geqq \|T\alpha\| > \|\alpha\| - 2\varepsilon$.

Q.E.D.

COROLLARY 4. *If X is a separable (AL)-space, then there is a linear positive isometry from X^* into m which carries 1 to 1.*

PROOF. Let $T: l^1 \to X$ be as above. $T^*: X^* \to m$ is a linear isometry. Since $1 \in X^*$ is defined by $1(x) = \|x_+\| - \|x_-\|$, $(T^*1)(\alpha) = 1(T\alpha) = \|(T\alpha)_+\| - \|(T\alpha)_-\| = \|T\alpha_+\| - \|T\alpha_-\| = \|\Sigma_{\alpha_i>0}\alpha_i y_i\| - \|\Sigma_{\alpha_i<0}\alpha_i y_i\| = \Sigma_{\alpha_i>0}\alpha_i + \Sigma_{\alpha_i<0}\alpha_i = \Sigma\alpha_i = 1(\alpha)$.

Q.E.D.

REMARKS. 1) The necessity of the existence of an extension operator is demonstrated by:

PROPOSITION 5. *If $S \subset \beta N$ and $C(S)$ is isomorphic to a subspace of $C(\beta N)$, then S is Stonian. In particular, $C(\beta N \backslash N)$ is not isomorphic to a subspace of $C(\beta N)$.*

PROOF. Following Rosenthal [13], S is said to satisfy the countable chain condition (CCC) if every disjoint family of non empty open sets in S is at most countable. By [13, p. 229], S satisfies the CCC if and only if $C(S)$ contains no isomorph of $c_0(\Gamma)$ for uncountable Γ. Since βN satisfies the CCC, if $C(S)$ is isomorphic to a subspace of $C(\beta N)$ then S too satisfies the CCC. If $S \subset \beta N$ then S is an F-space. By [14] p. 19, every F-space satisfying the CCC is Stonian.

Q.E.D.

2) If $C(S) \subset C(T)$, Theorem 1 establishes the existence of the following diagram:

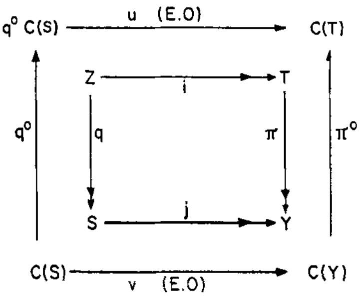

Fig. 1

where u, v are norm-1 linear extensions. The diagram would have looked prettier if we could replace $q^0C(S)$ by $C(Z)$. The following simple example shows that a linear extension $u: C(Z) \to C(T)$ may fail to exist if Z is the Holsztynski space constructed in the proof of theorem 1: $T = \beta N$, $S = \{s\}$, $(\phi c)(n) = c(1-(1/n))$, $(\phi c)(p) = c$ for $p \in \beta N \setminus N$. One easily checks that Z in this case is $\beta N \setminus N$, which admits no linear extension to βN.

PROPOSITION 6. *Suppose $C(S)$ is isometric to a subspace of $C(T)$. Then in each of the following cases there are a subspace Z of T admitting a norm-1 linear extension and a quotient map of Z onto S:*

a) *T is metrizable.* b) *S is Stonian.* c) *T is Stonian.*

PROOF. a) Is immediate from Theorem 1 and the fact that in a metric compact T every closed subspace admits a norm-1 linear extension [10].

b) Is immediate from Corollary 2.

c) Follows from b) and the following result of H. B. Cohen [4]: For every Banach space B there are a Stonian space S_B and a linear isometry $u_B : B \to C(S_B)$ such that, given a linear isometry $v: B \to X$ (X an arbitrary Banach space), any linear w: $C(S_B) \to X$ satisfying $wu_B = v$ is an isometry. If $B = C(S)$ and S is Stonian, then this implies that S_B must be homeomorphic to S. If $B = C(S)$, S any, then S_B turns out to be the Gleason space of S [8].

In our case, let $B = C(S)$ and $v: B \to C(T)$ the given isometry. Then vu_B^{-1} is a linear isometry from $u_BC(S)$ to $C(T)$. Since T is Stonian, $C(T)$ has the extension property ([5], p. 94) and vu_B^{-1} can be extended to a norm-1 w: $C(S_B) \to C(T)$. By the characterizing property of S_B, w is an isometry. By b) there are a subspace Z of T admitting a norm-1 linear extension and a quotient map of Z onto S_B. Combined with the Gleason quotient map of S_B onto S [8], we get a quotient map of Z onto S. Q.E.D.

Important generalizations of the class of metrizable compacts are the class of *dyadic spaces* — the continuous images of generalized Cantor sets 2^m (2 stands for the two-point space, m for an infinite cardinal) and the class of *Dugundji spaces* [12] — those compact spaces S which admit a regular norm-1 extension in every embedding. The following proposition relates two open problems concerning these classes and the problem of extending Proposition 6.

PROPOSITION 7. *At least one of the following is true:*

a) *There is a nondyadic Dugundji space (cf.* [12], *problem* 14).

b) *There are no dyadic T and nondyadic compact S with $C(S) \subset C(T)$.*

c) *There exists a (nondyadic) compact S with $C(S) \subset C(2^m)$ such that S is not a quotient space of any subspace of 2^m admitting a norm-1 extension.*

PROOF. Suppose $C(S) \subset C(T)$, $T = \pi(2^m)$, S nondyadic. Then $C(S) \subset C(2^m)$, so we can assume $T = 2^m$. If $S = qZ$, then Z is nondyadic. It remains to show that if Z is a subspace of 2^m admitting a norm-1 linear extension u: $C(Z) \to C(2^m)$ then Z is Dugundji. This will follow from [12, p. 35] if we show that there is a regular extension for some embedding of Z in 2^m. If $m \leqq \aleph_0$ then Z is metrizable so that this follows from Michael's theorem. If $m > \aleph_0$, then $W = \{t \in 2^m;\ u1_z(t) = 1\}$, as a closed G_δ-subset of 2^m, is homeomorphic with 2^m [7].

3) Another generalization of the Banach-Stone theorem was given in [1] and [3]: If there is an isomorphism ϕ of $C(S)$ *onto* $C(T)$ and $\|\phi\|\,\|\phi^{-1}\| < 2$, then S is homeomorphic to T. It is not known whether one can replace 2 by 3. Using the technique of [1], one can show that if there is an isomorphism ϕ of $C(S)$ *into* $C(T)$ with $\|\phi\|\,\|\phi^{-1}\| < 2$, then S is a quotient space of a subspace of T. Questions which arise naturally are: a) If ϕ: $C(S) \to C(T)$ is an isomorphism into, do there exist a quotient Y of T and a linear extension from $C(S)$ to $C(Y)$? Suppose $\|\phi\|\,\|\phi^{-1}\| < 3$, or even $\|\phi\|\,\|\phi^{-1}\| < 2$? b) If ϕ: $C(S) \to C(T)$ is an isomorphism into with $\|\phi\|\,\|\phi^{-1}\| < 3$ (or even: $\|\phi\|\,\|\phi^{-1}\| < 2$), is $C(S)$ isometric to a subspace of $C(T)$?

3. Quotient maps of $C(S)$ onto $C(T)$.

THEOREM 8. *If τ: $C(S) \to C(T)$ is an exact quotient map (i.e. $\|g\| = \min\{\|f\|;\ f \in C(S),\ \tau f = g\}$ for all $g \in C(T)$), then there are a subspace Z of S, a quotient map q of Z onto T, a norm-1 projection w of $C(Z)$ onto $C(T)$ and $e \in C(S)$ such that $\tau(ef) = wi^0 f$ for every $f \in C(S)$, where i is the embedding of Z in S. If τ is regular, one can take $e = 1$, w regular.*

PROOF. Choose $e \in C(S)$ such that $\| e \| = 1$ and $\tau e = 1$. Denote $S' = \{s \in S; |e(s)| = 1\}$. If $f \in C(S)$ vanishes in a neighborhood of S' then we have $\| 1 + \lambda \tau f \| \leqq \| e + \lambda f \| = 1$ for small $|\lambda|$, which cannot hold in all directions unless $\tau f = 0$. Suppose now f vanishes on S', then, given any $\varepsilon > 0$, f agrees with a function f' of norm $< \varepsilon$ in a neighborhood of S', and by the above $\| \tau f \| = \| \tau f' + \tau(f - f') \| = \| \tau f' \| < \varepsilon$, and $\tau f = 0$ for every $f \in C(S)$ vanishing on S', i.e. $\tau = \tau' i'^0$, where i' is the embedding of S' in S, and τ' is an exact quotient map of $C(S')$ onto $C(T)$. $\psi(h) = \tau'(e'h)$, where $e' = i'^0 e$, is a *regular* exact quotient map of $C(S')$ onto $C(T)$ — in particular ψ is positive. Define, for $t \in T$:

$$\rho(t) = \bigcap_{\substack{\psi h \in P(t) \\ \|h\| = 1}} \{s \in S'; h(s) = 1\},$$

where $P(t)$ is defined as in the proof of Theorem 1. We show first that $\rho t \neq \varnothing$: If $\psi h_i \in P(t)$ and $\|h_i\| = 1$ $(i = 1, \cdots, n)$ then, since $1/n \ \sum_{i=1}^n \psi h_i(t) = 1$, there is some $s \in S'$ with $n^{-1} \sum_{i=1}^n h_i(s) = 1$, i.e. $h_i(s) = 1$ for $i = 1, \cdots, n$. This shows that the intersected family has the finite intersection property and, by compactness, has a non void intersection.

If $t_1 \neq t_2$, take $g \in P(t_i)$ with $-g \in P(t_2)$. Suppose $s \in \rho t_1 \cap \rho t_2$ and take $h \in C(S')$ with $\| h \| = 1, g = \psi h$. Then $h(s) = 1$ since $\psi h \in P(t_1)$ and $s \in \rho t_1$, and $h(s) = -1$ since $\psi(-h) \in P(t_2)$ and $s \in \rho t_2$. This is impossible, so that $\rho t_1 \cap \rho t_2 = \varnothing$ for $t_1 \neq t_2$.

If K is a closed subset of T, then ρK is closed in S': Suppose $\rho t_\alpha \ni s_\alpha \to s$, $t_\alpha \in K$. We may assume $t_\alpha \to t \in K$. Let $h \in C(S')$, $\| h \| = 1$, $\psi h \in P(t)$. Then $\psi h \in P(t_\alpha)$ hence $h(s_\alpha) = 1$ for $\alpha > \alpha_0$ and $h(s) = 1$, i.e. $s \in \rho t$. Thus $q = \rho^{-1}$ is continuous, and T is the quotient space of the compact $Z = \rho T$.

We show now that ψh is determined by the values of h on Z. If $s \in S \backslash Z$ and $t \in T$, there is some $h_t \in C(S')$ with $\psi h_t \in P(t)$, $\| h_t \| = 1$ and $h_t(s) < 1$. Replacing h_t by $(h_t)_+$ we may assume also that $h_t \geqq 0$. If V_t is a neighborhood of t on which $\psi h_t = 1$, take a finite cover for T: $V_{t_1}, \cdots, V_{t_n}$, and let $h = \max\{h_{t_i}; i = 1, \cdots, n\}$. Then $\| h \| = 1$ while $\psi h \geqq \max \psi h_{t_i} = 1$, i.e. $\psi h = 1$ while $h(s) < 1$.

For a compact K in $S' \backslash Z$ we take, for each $s \in K$, such $h_s \in C(S')$ with $h_s \geqq 0$, $\| h_s \| = 1$, $\psi h_s = 1$ and $h_s(s) < 1 - \varepsilon_s$ $(\varepsilon_s > 0)$. Let $W_s = \{r \in S'; h_s(r) < 1 - \varepsilon_s\}$. Choose a finite cover for K: $W_{s_1}, \cdots, W_{s_n}$, and let $h_K = \max\{h_{s_i}; i = 1, \cdots, m\}$. Then $h_K \in C(S')$, $\| h_k \| = 1$, $h_K \geqq 0$, $\psi h_k = 1$ while $h_K(K) < 1 - \varepsilon_K$ where $0 < \varepsilon_K = \min\{\varepsilon_{s_i}; i = 1, \cdots, m\}$. Define now $H_K = \varepsilon_K^{-1} \min(1 - h_K, \varepsilon_K)$. Then $H_K \in C(S')$, $0 \leqq H_K \leqq 1$, $H_K(K) = 1$ while $\psi H_K = 0$.

Given $h \in C(S')$, $0 \leqq h \leqq 1$ and $h(Z) = 0$, take $K = \{s \in S'; h(s) \geqq 1/2\}$ and let $h' = \max(h - 2^{1-}H_K, 0)$. Then $\psi h' \geqq \psi(h - 2^{-1}H_K) = \psi h$, while $0 \leqq h' \leqq 1/2$. We have succeeded to halve the norm of h without affecting ψh. A successive application of this procedure shows that $\psi h = 0$. This shows that $\psi = wi''^0$, where i'' is the embedding of Z in S'. Thus, for $f \in C(S)$: $\tau(ef) = \tau'(i'^0(ef)) = \tau'((i'^0 e)(i^0 f)) = \tau'(e' \cdot i^0 f) = \psi(i^0 f) = w(i''^0 i'^0 f) = w((i'i'')^0 f) = w(i^0 f)$, where $i = i'i''$ is the embedding of Z in S.

It remains to prove that w is a projection, i.e. $wq^0 g = g$ for all $g \in C(T)$. For a compact K in T and a neighborhood U of K, let $g_{KU} \in C(T)$ satisfy $0 \leqq g_{KU} \leqq 1$, $g_{KU} = 1$ on a neighborhood of K, $g_{KU} = 0$ on a neighborhood of $T \backslash U$. Let $h_{KU} \in C(Z)$ satisfy $\| h_{KU} \| = 1$, $h_{KU} \geqq 0$, $wh_{KU} = g_{KU}$. Then $h_{KU} = 1$ on ρK and $h_{KU} = 0$ on $\rho(T \backslash U)$. Define, for $g \in C(T)$, $\| g \| = 1$ and every n:

$$A_i^n = \{t; g(t) \geqq \frac{i}{n}\},\ B_i^n = \{t; g(t) > \frac{i-1}{n}\}\ (i = 1, \cdots, n),$$

$$h_i^n = h_{A_i^n B_i^n},\ h_n = \frac{1}{n} \sum_{i=1}^{n} h_i^n.$$

If

$$\frac{j}{n} \geqq g(t) \geqq \frac{j-1}{n},$$

then

$$|(wh_n - g)(t)| = \left|\frac{1}{n} \sum_{i=1}^{n} g_{A_i^n B_i^n}(t) - g(t)\right| = \left|\frac{j-1}{n} + \frac{1}{n} g_{A_j^n B_j^n}(t) - g(t)\right| \leqq \frac{1}{n}$$

and

$$|(h_n - q^0 g)(\rho t)| = \left|\frac{1}{n} \sum_{i=1}^{n} h_i^n(\rho t) - g(t)\right| = \left|\frac{j-1}{n} + \frac{1}{n} h_j^n(\rho t) - g(t)\right| \leqq \frac{1}{n}.$$

Therefore $h_n \to q^0 g$ uniformly on Z while $wh_n \to g$ uniformly on S, which shows that $g = wq^0 g$. Q.E.D.

COROLLARY 9. *If $\tau: C(S) \to C(T)$ is an exact quotient map and T is Stonian, then T is homeomorphic to a subspace of S.*

PROOF. Immediate from Theorem 8 and the projectivity of Stonian spaces.

REMARK. A quotient map τ of $C(S)$ onto $C(T)$ need not be exact. Take any infinite compact S, let $T = \{t\}$, $\tau f(t) = \phi(f)$ where $\phi \in C(S)^*$ does not attain its supremum on the unit ball. One may ask: Suppose $C(T)$ is a quotient space of $C(S)$, is it also an exact quotient space?

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